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Mean-field theory with symmetry breaking for the critical properties of the q-state Potts model

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Abstract. We show that the dependence of the critical temperature T_c on the space dimension d of the q-state Potts model can be fitted by straight lines, with slopes predicted by a long-range interaction mean-field theory with broken permutational symmetry: $\mathcal{S}_q \supset \mathcal{S}_m \otimes \mathcal{S}_{q-m}$. By means of phenomenological arguments we 'reconstruct', in terms of the above mean field, the (d, T_c) plots with a high degree of accuracy. The model suggests a factorisation of the partition function near or at the critical point. Predictions, implied by this ansatz, are given on the behaviour of the specific heat exponent α and the latent heat L which are corroborated by 'data' presently available in the literature: $q = 4$ with $2.5 \leqslant d \leqslant 6$ for α, and $q \geqslant 4$ with $1 \leqslant d \leqslant 2$ for L.

1. Introduction

When dealing with critical exponents of phase transitions, the availability of precise empirical data and numerical work suggested a scaling behaviour near the critical point. Later on, the observed universality was justified rigorously by renormalisation group techniques which provided a better understanding of the dependence of the values of the critical exponents on the space dimension d and the dimension n of the order parameter (see e.g. Pfeuty and Toulouse 1977). Guided by the history of the critical exponents, it seems to us that a search for empirical relations for the critical temperature T_c might be fruitful for further theoretical developments. It was in this spirit that we analysed in a previous note (Cocho *et al* 1982, which we shall refer to as I): the dependence of T_c on d for the Ising model on hypercubic lattices. The available analytical and numerical data for T_c turned out to be fitted with a remarkable accuracy by two straight lines in the (d, T_c) plane, suggesting the presence of a 'restricted' form of self-duality for the Ising model when $d \leqslant 4$ and a cluster approximation mean-field (MF) behaviour with anisotropic interactions for $d > 4$.

In this work we extend the above analysis to the q-state standard Potts model (Potts 1952, Wu 1982) in a d-dimensional hypercubic lattice ($q = 2$ corresponding to the Ising model), and produce a 'mean field theory with symmetry breaking' (MFSB) which exhibits a wide variety of the critical properties of the Potts model. In § 2 we present some known (Wu 1982) results of the critical behaviour of the Potts model from which we develop in § 3 the above mentioned MFSB. In § 4 we compare the results from this MFSB with the data of § 2, analysing the behaviour of the Potts model for different q, and obtain a formula for the dependence of the critical temperature T_c on d. The line of reasoning behind this formula is then extended to the analysis of

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the behaviour of the critical exponent for the specific heat and of the latent heat. The most significant result of our work, which is developed in § 4, is that the critical behaviour of the q-state Potts model in dimension d may be accurately described by partitioning the space into subspaces of various dimensions and using a different mean-field model in each subspace. All of these coexisting mean-field models are assumed to have the same critical temperature. For given q, the critical temperature is assumed to be a continuous piecewise linear function of the dimensionality, the slope of which is determined in each range of d by only one of the coexisting models. The model chosen is said to be dominant in that range.

Finally, in § 5 we summarise and discuss our results pointing out the predictive component of the MFSB and the implications of a conjecture presented there.

2. Summary of numerical and analytic data

The Potts model has been a subject of increasing research interest in recent years (cf Wu 1982 for a comprehensive updated review). In this paper we shall be refering to the q-state standard Potts model (Wu 1982) which is described by the 'Hamiltonian'

$$\mathscr{H} = -\varepsilon \sum_{\langle i,j \rangle} \delta_{\sigma_i \sigma_j}, \tag{1}$$

where $\sigma_i (\sigma_i = 1, 2, \ldots, q)$ is a spin variable on the site of a d-dimensional hypercubic lattice which can assume q-values, ε is a coupling constant, and the sum runs over all nearest-neighbour pairs, each pair is counted only once. Whenever two nearest-neighbours are in the same state there is an interaction energy of $-\varepsilon$, otherwise it is zero.

Following the phenomenological approach set out in I, we shall firstly concentrate on the dependence of $K_c^{-1} \equiv k_B T_c / \varepsilon$ (k_B = Boltzmann's constant) on d. (Note that

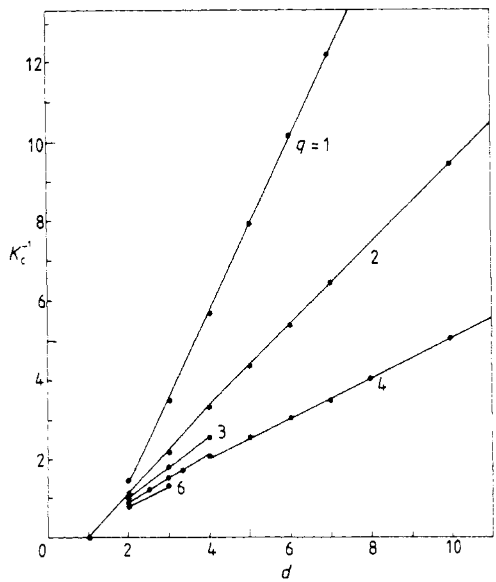

Figure 1. Plot of the critical values $K_c^{-1}(d)$ for various q-values of the Potts model taken from tables II and III of Wu (1982) and table I of Cocho *et al* (1982). The straight lines are to be taken only as a guide to the eye.

K_c^{-1} corresponds to $2X_c^{-1}$ of I, i.e. $2J=\varepsilon$). For this purpose we have plotted in figure 1 the values currently available in the literature (Wu 1982) of K_c^{-1} for different d and q, estimated either by series, Monte Carlo and renormalisation group analysis, or from duality considerations (when $d=2$, K_c^{-1} can be obtained from duality arguments for arbitrary q; see Potts 1952 and Wu 1982).

From figure 1 we see that, just as in the Ising case, the dependence of T_c on d for a given q appears to be linear (the lines sketched in figure 1 are to be taken only as a guide to the eye).

In I we remarked that for $q=2$, below $d_c=4$ the linear dependence could be understood in terms of a 'restricted' self-dual behaviour, while for $d>4$, where the transition is MF-like, the linear dependence corresponded to an anisotropic MF cluster approximation behaviour. For general q, the value $d_c(q)$, above which a mean-field dynamic prevails, is expected to fall on the scheamatic curve of figure 2 (see Nienhuis *et al* 1981 and Wu 1982). From the comments presented in I, it follows that for $d>d_c(q)$ some type of MF theory of an anisotropic character might describe the T_c behaviour.

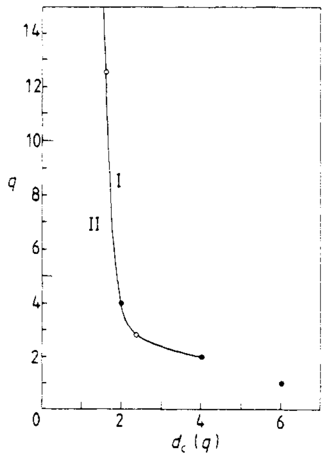

Figure 2. Schematic plot of the critical dimension $d_c(q)$ beyond which, for a given q, a mean-field behaviour prevails (Wu 1982). For $q>2$, $d_c(q)$ is the value of d at which the Potts transition changes from continuous to first-order (Nieunhuis *et al* 1981). The full circles are known results taken from Wu (1981) while the open circles are results of the variational renormalisation group calculations of Nienhuis *et al* (1981).

3. Mean-field theory with symmetry breaking

Several MF theories have been proposed for the study of the q-state Potts model (see Wu 1982), in this section we shall develop a MF treatment which in an extension of the one discussed in the review by Wu (1982).

We start from the infinite-range interaction MF Hamiltonian (Husimi 1953, Temperley 1954, Kac 1968)

$$\mathscr{H}=-(1/N)\gamma\varepsilon\sum_{i<j}\delta_{\sigma_i\sigma_j}, \qquad (2)$$

for a system of N spins, each of which interacts with the other $N-1$ spins via an equal strength of $\gamma\varepsilon/N$, γ being the coordination number of the lattice ($\gamma=2d$ for hypercubic lattices). Following Wu (1982), if we define χ_i as the fraction of spins that are in the spin state $i=1, 2, \ldots, q$, subject to the constriction:

$$\sum_i \chi_i = 1, \tag{3}$$

then to leading order in N, the energy and entropy per spin are

$$E/N = -\tfrac{1}{2}\gamma\varepsilon \sum_i \chi_i^2, \tag{4}$$

$$S/N = -k_{\mathrm{B}} \sum_i \chi_i \ln \chi_i, \tag{5}$$

and the free energy per spin A, is given by the expression

$$A/k_{\mathrm{B}}T = \sum_i (\chi_i \ln \chi_i - \tfrac{1}{2}\gamma K \chi_i^2), \tag{6}$$

where $K=\varepsilon/k_{\mathrm{B}}T$, with $\varepsilon>0$ for a ferromagnetic-type interaction.

Since from I we expect our MF approximation to be of an anisotropic nature, we look for a solution in the form of

$$\begin{aligned} \chi_a &= (1/q)[1+(q-m)s], \qquad & a&=1,2,\ldots,m \\ \chi_b &= (1/q)[1-ms], \qquad & b&=m+1,\ldots,q \end{aligned} \tag{7}$$

where $1 \leqslant m \leqslant q$ is a positive integer which acts as a symmetry breaking factor and the 'order parameter' s, $0 \leqslant s \leqslant 1/m$ is to take the value s_0 which minimises the free energy. We shall refer to the above choice of spin fractions as the $[m, q-m]$ partition. (The infinite range MFSB with the $[1, q-1]$ partition corresponds to the cluster type MF mentioned in I; cf Nakanishi and Stanley 1981.)

From (6) and (7) we have

$$\begin{aligned} \Delta(s) &\equiv \frac{1}{k_{\mathrm{B}}T}[A(s)-A(0)] \\ &= (m/q)[1+(q-m)s]\ln[1+(q-m)s] + [(q-m)/q][1-ms]\ln[1-ms] \\ &\quad -\tfrac{1}{2}\gamma K[m(q-m)/q]s^2 \end{aligned} \tag{8}$$

and

$$\Delta'(s) = [m(q-m)/q](\ln\{[1+(q-m)s]/(1-ms)\} - \gamma K s). \tag{9}$$

At the phase transition, whenever $q>1$, s and K take the MF critical values $s_{\mathrm{c}}^{\mathrm{MF}}$ and $K_{\mathrm{c}}^{\mathrm{MF}} \equiv \varepsilon/k_{\mathrm{B}}T_{\mathrm{c}}^{\mathrm{MF}}$ which satisfy (Wu 1982)

$$\Delta'(s_{\mathrm{c}}^{\mathrm{MF}}) = \Delta(s_{\mathrm{c}}^{\mathrm{MF}}) = 0, \tag{10}$$

i.e.

$$s_{\mathrm{c}}^{\mathrm{MF}} = (q-2m)/m(q-m) \tag{11}$$

and

$$\gamma K_{\mathrm{c}}^{\mathrm{MF}} = [2m(q-m)/(q-2m)]\ln[(q-m)/m]. \tag{12}$$

Expression (12) gives us the dependence of the MF critical temperature T_c^{MF} on d, i.e.

$$T_c^{MF} k_B/\varepsilon \equiv (K_c^{MF})^{-1} = (q-2m)d/m(q-m)\ln[(q-m)/m]. \quad (13)$$

A further quantity which can be calculated directly from (11), (12) and (5) is the MF latent heat per spin L^{MF}:

$$L^{MF} = \varepsilon(q-2m)^2 d/qm(q-m). \quad (14)$$

A few remarks concerning equations (11) to (14) are appropriate at this stage:

(i) If $m \to \frac{1}{2}q$, then from (11), (12) and (5) we have that $s_c^{MF} \to 0$, $\gamma K_c^{MF} \to q$ and $L^{MF} \to 0$.

(ii) For the $q = 1$ case, we can take equations (8) and (9) with $q = 1+\varepsilon$ and $m = 1$, i.e. the formal partition $[1, 1+\varepsilon]$; impose the criticallity condition $\Delta(s_c^{MF}) = \Delta'(s_s^{MF}) = 0$ and at the end take the limit $\varepsilon \to 0$. The final result being, just as in the above case, $s_c^{MF} = 0$ and $\gamma K_c = q = 1$.

(iii) Cases (i) and (ii) are MF continuous transitions; otherwise, i.e. $m \neq \frac{1}{2}q$ and $q \neq 1$, we have MF first-order transitions.

(iv) Equations (12), (13) and (14) are invariant under the transformation $[m, q-m] \to [q-m, m]$ i.e. the MF critical temperature and latent heat of the partitions $[m, q-m]$ and $[q-m, m]$ are the same. When dealing with s_c^{MF}, the above transformation sends s_c^{MF} to $(-s_c^{MF})$, as expected.

(v) The following scaling relations hold

$$\begin{aligned} s_c^{MF} &= q^{-1} f(m/q), & K_c^{MF} &= qh(m/q), \\ L^{MF} &= q^{-1} w(m/q), & L^{MF} K_c^{MF} &= p(m/q), \end{aligned} \quad (15)$$

where the functional form of f, h and w follows directly from equations (11), (12) and (14) respectively, and $p = hw$.

4. Dependence of critical properties on space dimensionality

So far we have established a MF theory with a symmetry breaking factor m, and derived within this approximation expressions for the critical quantities s_c^{MF}, T_c^{MF} and L^{MF}. In order to obtain information on the critical properties for the 'exact' (fully interacting) q-state Potts model, we shall proceed along the lines set out in I.

4.1. Critical temperature

In I we showed that the slope of T_c as a function of d, for $d \geq d_c(2) = 4$, is remarkably well approximated by an anisotropic MF theory. On the other hand, we know from equation (13) the slope of T_c^{MF}, for the general q-state Potts model, as a function of d for every partition $[m, q-m]$. In the following, we shall therefore discuss how much information on the critical behaviour of the Potts model we can extract from the MFSB, by considering the case of each q (for which we have found in the literature numerical values for T_c) separately.

4.1.1. $q = 2$. This is the Ising model studied in I. For this case we know that, whenever $d \geq d_c(2) = 4$, T_c can be fitted by the expression (see equation (12) of I)

$$T_c k_B/\varepsilon = K_c^{-1} = 3/\ln(1+\sqrt{2}) + (d-4) \equiv [3/4\ln(1+\sqrt{2})]d_c(2) + (d - d_c(2)), \quad (16)$$

where the first term in the last member of (16) can be thought of as a contribution from the non-mean-field region (region II of figure 2) expressed in terms of an 'effective mean-field' (EMF) type coupling $3\varepsilon/4 \ln(1+\sqrt{2})$, and the second term is the contribution from the mean-field region (region I of figure 2).

In the context of the MFSB the following observations can be made.

(i) We expect it to describe the critical behaviour of the system in region I of figure 2, i.e. above $d = d_c(2) = 4$.

(ii) The only partition is [1, 1], which corresponds to a MF continuous transition (recall remarks (i) and (iii) of § 3).

(iii) When dealing with continuous phase transitions, it is common knowledge (see Pfeuty and Toulouse 1977) that there is a dimension d_J such that Josephson's scaling law, $\nu d = 2 - \alpha$, is obeyed by exponents with their 'classical' or MF values. Given the above remarks, the MFSB should hold for $d \geq d_J(q)$, where $d_J(q) = 4$ for $q > 1$.

From these observations and remark (i) of § 3 ($\gamma K_c = q$ for $m = \frac{1}{2}q$) we expect, for $d > 4$, the slope of K_c^{-1} as a function of d, to be 1; which is in agreement with (16) and is plotted in figure 3.

4.1.2. $q = 4$. Just as in the $q = 2$ case, we shall be concerned with the behaviour of the Potts model in region I of figure 2, i.e. for $d \geq d_c(4) = 2$. Since for $q = 4$ we have the two distinct partitions: [1, 3] and [2, 2], there seems to be some ambiguity in the choice for the slope of K_c^{-1} given by equation (13). At first sight, the [1, 3] partition seems to be the best choice since the transition which will take over is the first transition crossed when one comes from high temperature. However, it has been argued (de Magalhães and Tsallis 1981) that for large d the Potts model seems to approach a continuous transition behaviour. Since $d_J(4) = 4$ we might expect that the [2, 2] partition (corresponding to a continuous phase transition) should contribute to the phase transition when $d > d_J(4)$. Our results of §§ 4.2 and 4.3 as well as the comparison with the available numerical and recent renormalisation group calculations (see Indekeu *et al* 1982) suggest that this contribution is dominant in a subspace of dimension $d^1 = d - d_J(4)$, with $d > d_J$; the main point being that we shall envisage our system as composed of a 'mixture' of coexisting MF theories (corresponding to different partitions and the EMF) which hold in subspaces of smaller dimensionality and that come to criticality at the same temperature. We shall therefore propose the [1, 3] slope of $2/3 \ln 3$ (cf equation (13)) from $d_c(4) = 2$ to $d_J(4) = 4$, and the [2, 2] slope of $\frac{1}{2}$ for $d \geq d_J(4) = 4$.

The starting point in figure 3 for the [1, 3] branch is known from the duality relation (Potts 1952 and Wu 1982).

$$K_c(q) = \ln(1+\sqrt{q}) \tag{17}$$

valid for $d = 2$ and aribtrary q. For the [2, 2] branch we check that the $d = 4$ starting point satisfies the requirement that any MF critical temperature is an upper bound to the 'true' critical temperature (this has been rigorously proved for the continuous transition of multicomponent Heisenberg-type classical ferromagnets by Simon (1980) and Brydges *et al* (1982)). The broken line, corresponding to the [2, 2] partition, traced in figure 3 for $q = 4$ and $d \geq 4$ is the MFSB least upper bound. For this case, the upper bound is a remarkably good fit to the 'true' values of T_c.

Summing up, we have for $q = 4$, using equation (17) and (13) the following formula

$$K_c^{-1}(d) \leq 1/\ln 3 + (2/3 \ln 3)(d-2), \qquad 2 \leq d < 4 \tag{18}$$

and

$$K_c^{-1}(d) \leqslant 1/\ln 3 + 4/3 \ln 3 + \tfrac{1}{2}(d - 4 - \xi), \qquad d \geqslant 4 \tag{19}$$

with $\xi = 2/\ln 3 + 8/3 \ln 3 - 4$.

We have written (18) and (19) so as to emphasise the following general behaviour

$$K_c^{-1}(q, d) \leqslant \sum_{i=0}^{n(q,d)} W_{i+1}(q)[d_{i+1}(q) - d_i(q) - \xi_{i+1}(q)] + W_{n(q,d)+2}(q)[d - d_{n(q,d)+1}(q) - \xi_{n(q,d)+2}(q)], \tag{20}$$

where $n(q, d)$ is a non-negative integer, $d_{i+1}(q) > d_i(q)$, $d \geqslant d_{n(q,d)+1}(q)$ and $d_0(q) \equiv 0$.

For the $q = 4$, $d > 4$ case, $n(4, d) = 1$; the $W_1 = 1/2 \ln 3$ term with $d_1(4) = d_c(4) = 2$ and $\xi_1(4) = 0$ corresponds to the EMF behaviour mentioned in § 4.1.1; the $i = 1$ term given by $W_2(4) = 2/3 \ln 2$, $d_2(4) = d_J(4) = 4$, $\xi_2(4) = 0$, is the [1, 3] first-order phase transition contribution; and the $n(4, d) + 2 = 3$ term is the [2, 2] continuous transition contribution with $W_3(4) = \frac{1}{2}$ and $\xi_3(4) = \xi$ of equation (19).

Notice that if we look at the plot of (18) and (19) taken as equalities, we observe that there is a remarkable agreement with the available numerical data.

4.1.3. $q = 3$. For this case, we only have the [1, 2] first-order partition and $d_c(3) \approx 2.2$ (Nienhuis *et al* 1981). We therefore expect a straight line (full line in figure 3) of slope $1/2 \ln 2$ in the $d \geqslant d_c(3)$ region (region I of figure 2) lying below the MF upper bound $K_c^{-1} = d/2 \ln 2$. In figure 3 we have plotted this upper bound (broken line) and shown that the available data conform with the above predictions, however, more numerical data are necessary in order to check the behaviour predicted by the MFSB.

4.1.4. $q = 6$. When $q = 6$ the Potts MFSB has three distinct partitions: [1, 5], [2, 4] and [3, 3]. Since $d_c(6) < 2$ (Nienhuis *et al* 1981), the exact $d = 2$ result falls within the MF region I of figure 2. Because of the upper bound property of the MF critical temperatures, the [1, 5] branch is the only one that can pass through the dual point. An analysis similar to the $q = 4$ case leads us to expect that the [3, 3] branch will hold for $d > d_J(6) = 4$.

In figure 3 we have drawn the three branches for the $q = 6$ case. The [1, 5] branch (full line) gives a slope which agrees with available numerical data. The [2, 4] and [3, 3] branches (broken lines) are proposed as upper bound MF results. Since we have so far found no *a priori* criteria for the determination of the position of the discontinuity between the [1, 5] and [2, 4] branches, in figure 2 this discontinuity has been placed arbitrarily.

4.1.5. $q = 1$. This is a limiting case which corresponds to the percolation problem. From remark (ii) of § 3 we know that within the MF approximation it corresponds to a continuous transition and from remark (i) of the same section we have

$$K_c^{-1} = 2d. \tag{21}$$

Since $d_c(1) = 6 = d_J(1)$ (Toulouse 1974, Pfeuty and Toulouse 1977, Kirkpatrick 1976, Gaunt *et al* 1976) we expect (21) to be an upper bound for $K_c^{-1}(d)$ for all d, and 2 to be a good approximation for the slope above six dimensions. In figure 3, both assumptions are shown to hold; however, more numerical data would provide a better check. Notice that for $d \leqslant 6$ the EMF behaviour hypothesis is strongly corroborated (cf figure 1 and the chain line on figure 3).

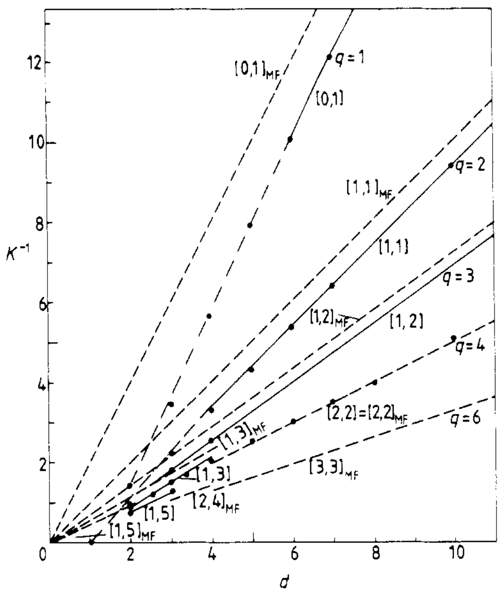

Figure 3. Plot of the critical values of $K^{-1} \equiv k_B T/\varepsilon$ as a function of d for several q values of the Potts model. The broken lines (denoted by $[m, q-m]_{\mathrm{MF}}$) correspond to the mean-field behaviour of the MFSB theory $[m, q-m]$ partition given by equation (13). These lines are upper bounds to the 'true' critical temperature. The full lines, labelled by $[m, q-m]$, are the predictions for the behaviour of the 'true' critical values of $K_c^{-1}(d)$ given by the MFSB via the phenomenological arguments of § 4. (The notation $[0, 1]$ stands for $\lim_{\varepsilon\to 0}[\varepsilon, 1-\varepsilon]$.) The chain lines correspond to the conjectured effective mean-field (EMF) behaviour. For $q=2$ this line is given by equation (4) of Cocho *et al* (1982). The $q=1$ chain line is only given as a guide to the eye. The full circles are the values of $K_c^{-1}(d)$ given in figure 1.

4.2. *Specific heat exponent* α

So far, in the analysis of the critical temperature behaviour, we have mentioned that for a given q-state Potts model, there are dimensional ranges where a partition $[m, q-m]$ appears to be dominant, and that some of these partitions correspond to first-order transitions, while others are continuous ones. On the other hand, there has been a constant preoccupation in the literature (Wu 1982, Nienhuis *et al* 1981, Livi *et al* 1983, de Magalhães and Tsallis 1981) concerning the type of phase transition of the q-state Potts model at a given dimension. We find, for example, that for the $q=4$ Potts model, which is expected to exhibit a MF type first-order transition with a latent heat above $d_c(4)=2$ (Wu 1982), values for the critical exponent α have been calculated numerically (Ditzian and Kadanoff 1979). We believe that this situation can be understood by extending the main phenomenological result of the critical temperature behaviour contained in equation (20) to other thermodynamic quantities, for example, the internal energy. The conjecture is that the internal energy can be written as a sum of the contributions from different MFSB partitions plus what we have called the EMF, i.e.

$$u(q, d) = \sum_{i=0}^{n(q,d)} u_{i+1}(q)[d_{i+1}(q) - d_i(q)] + u_{n(q,d)+2}(q)[d - d_{n(q,d)+1}(q)], \tag{22}$$

where $u(q, d)$ is the internal energy of the q-state Potts model and $u_{i+1}(q)$, $i \geq 1$, is the contribution from the ith partition 'weighted' by the range $[d_{i+1}(q) - d_i(q)]$ of dimensions where it is dominant, and $u_1(q)$ is the EMF region contribution (recall that $d_0(q) = 0$). The dimensions d_i, d, and the number $n(q, d)$ satisfy the conditions specified for equation (20).

The $q = 4$, $d > 4$, case would then be:

$$u(4, d) = u_1(4)[d_c(4) - 0] + u_2(4)[d_J(4) - d_c(4)] + u_3(4)[d - d_J(4)], \qquad (23)$$

the $u_1(4)$ term corresponding to the fully interacting system we have modelled by the EMF, $u_2(4)$ to the first order MF behaviour of the partition $[1, 3]$ and $u_3(4)$ to the continuous transition MF partition $[2, 2]$. We are not implying that for example, at $d = 5$, the system undergoes a MF continuous transition, but rather that equation (23) indicates an accumulative effect: the system retains some of the $u_1(4)$, $u_2(4)$ and $u_3(4)$ behaviour. From the definition of the specific heat, we would then expect that at $d = 5$ the $u_1(4)$ term would be the only one to give a contribution to the value of α, since $u_2(4)$ will give no divergence as $T \to T_c$ (first-order MF), and for $u_3(4)$ we have a jump discontinuity (MF continuous transition).

We can thus envisage the behaviour of the $q = 4$ Potts model for $d = 5$ as being described by a $[1, 3]$ MF Potts model in a 'two-dimensional subspace' 'coexisting' with an EMF which holds in another 'two-dimensional subspace' and a $[2, 2]$ MF model in the 'remaining one dimension'. All these mean-field behaviours coexist and enter criticality at the same temperature.

We are therefore stating that for $q = 4$ and $d \geq 2$, the specific heat exponent will take its $d = 2$ value, which has been conjectured to be $\frac{2}{3}$ (there is firm numerical support for this conjecture; Wu 1982). This behaviour is consistent with the numerical calculations of Ditzian and Kadanoff (1979), which give the mean-weighted average value of 0.662 ± 0.034 for $2.5 \leq d \leq 6$. Numerical results for α, for various values of d and q, would provide us with a testing ground for the behaviour implied by equation (23).

4.3. *Latent heat L*

For the latent heat the picture again is similar to that of equation (20):

$$L(q, d) = \sum_{i=0}^{n(q,d)} L_{i+1}(q)[d_{i+1}(q) - d_i(q)] + L_{n(q,d)+2}(q)[d - d_{n(q,d)+1}(q)], \qquad (24)$$

where the subscript i has the same role as in equations (20) and (23). L_{i+1} with $i \geq 1$ being the latent heat of the ith partition of the MFSB given by equation (14) divided by d, and $L_1(q)$ that of the EMF region. In equation (24), $n(q, d)$, d_i and d satisfy the requirements imposed for equation (20).

If we now look again at the $d = 5$, $q = 4$ case, we have that L $(4, 5)$ given by equation (24) reduces to

$$L(4, 5) = L_2(4)[d_J(4) - d_c(4)] = \tfrac{2}{3}\varepsilon \qquad (25)$$

since the continuous transition regions do not contribute to the latent heat. Notice that this line of reasoning provides an explanation for the presence of critical exponents in first-order ($L \neq 0$) transitions.

Fortunately, for the latent heat there is a testing ground for the conjecture behind equation (24) presently available in the literature on the Potts model: let us focus our attention on the region in figure 2 corresponding to $q \geq 4$ and $d \leq 2$. Since within this region, the points to the left of the curve sketched in figure 2 are expected to correspond to continuous transitions while those to its right to first-order transitions (Nienhuis *et al* 1981), then equation (24) for $q \geq 4$ and $d = 2$ reads:

$$L(q, 2) = L_2(q)[2 - d_1(q)], \tag{26}$$

where $d_1(q) = d_c(q)$ indicates the position of the above mentioned curve and $L_2(q)$ is the $[1, q-1]$ case of equation (14) divided by d. In equation (26) we are only left with the $L_2(q)$ term because the partition $[1, q-1]$ is the only one which satisfies the condition that the MF critical temperature is an upper bound to the 'true' temperature for $d = 2$. Also in (26) we have made use of $L_1(q) = 0$.

With equation (26) we can now determine the value of $d_1(q)$ because there is a closed expression for $L(q, 2)$ (Wu 1982). From equations (5.11), (5.12) and (5.15) of the review of Wu (1982)—there is a factor $\frac{1}{2}$ missing in expression (5.11) if we identify ε_2 with ε; (Wu, private communication)—we have:

$$L(q, 2) = \varepsilon(1 + q^{-1/2}) g(q) \tanh\tfrac{1}{2}\theta \equiv \varepsilon F(q) \tag{27}$$

where $\cosh\theta = \frac{1}{2}\sqrt{q}$, $\theta \geq 0$ and $g(q) = \Pi_{n=1}^{\infty}(\tanh n\theta)^2$.

Substituting (14) and (27) in (26) we obtain, for $q \geq 4$:

$$d_c(q) = d_1(q) = 2 - F(q)q(q-1)/(q-2)^2. \tag{28}$$

The curve corresponding to equation (28) (see figure 4) passes very near the points calculated by Nienhuis *et al* (1981) using renormalisation group techniques, and has the qualitative behaviour expected beforehand (i.e. the curve on figure 2). Notice that, $d_1(q) \to 1$ when $q \to \infty$, as predicted by Nienhuis *et al* (1981) and Livi *et al* (1983).

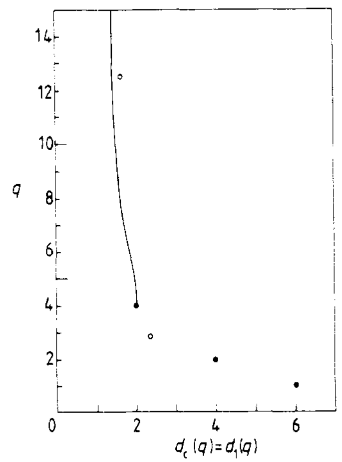

Figure 4. Plot of the values of $d_1(q) = d_c(q)$ for $q \geq 4$ predicted by equation (28). The full and open circles are the data plotted in figure 2.

Equation (28) is one of the main results of this paper. It gives support to the conjectures behind equations (20) (taken as an equality), (22) and (24), and is indicative of the advantages of the approach we have been following: our knowledge of $L(q, 2)$ allows us to determine $d_c(q)$ for all $d \leq 2$ and $q \geq 4$ (all this in terms of MF theories).

Again, numerical values for $L(q, d)$ and d_c(q) would be useful to give a more precise check on the validity of our assumptions.

5. Summary and conclusions

In this paper, by focusing our attention on the dependence of the critical temperature $K_c^{-1}(q, d)$ on space dimensionality d for the the q-state Potts model, we have observed from numerical data an essentially linear behaviour by parts (figure 1). We have constructed, prompted by a previous analysis for the Ising model (Cocho *et al* 1982), a MFSB that reproduces within the MF region (region I of figure 2), the slopes of $K_c^{-1}(q, d)$ for all q for which we have found numerical results (figure 3). By means of straightforward arguments we have established inequality (20), which taken as an equality gives an excellent fit for the numerical data. This phenomenological result has a *predictive* nature, at least as an upper bound.

If we analyse, with the above line of reasoning, the behaviour of the internal energy and latent heat (equations (22) and (24)), we can provide an explanation for the coexistence of critical exponents and latent heats, and of the dependence of the critical exponent α on dimensionality for the $q=4$ Potts model.

With the aid of the MFSB, we have been able to obtain the curve of $d_c(q)$ (equation (29)) for $q \geq 4$ and $d \leq 2$ from the values of the latent heat for $d=2$. This result gives support to the conjecture involved in equations (20), (taken as an equality), (22) and (24), and shows the calculational advantages of the approach we have adopted. Further numerical work is desirable as a check to the behaviour predicted by equation (28). The results so far obtained suggest that *near* or *at* the critical point the partition function, for the d-dimensional q-state Potts model, might factorise as a product of MF partition functions of lower dimensionality.

The main point we intend to put across in this paper is that near or at the critical temperature the behaviour of the exact, fully interacting system, can be remarkably well approximated in terms of MF theories which exhibit the simplicity of the description of the system under these conditions. Further analytical and numerical work is called for in order to strengthen or limit the validity of the points raised in this work.

References

Brydges D, Frölich J and Spencer T 1982 *Commun. Math. Phys.* **83** 123–50
Cocho G, Martínez-Mekler G C and Martínez-Enriquez R 1982 *Phys. Rev.* B **26** 2666–8
De Magalhães A C N and Tsallis C 1981 *J. Physique* **42** 1515–23
Ditzian R V and Kadanoff L P 1979 *J. Phys. A: Math. Gen* **12** L229–33
Gaunt D S, Sykes M F and Ruskin H 1976 *J. Phys. A: Math. Gen.* **9** 1899–911
Husimi K 1953 in *Proc. Int. Conf. on Theoretical Physics* ed H Yukawa (Tokyo: Science Council of Japan)
Indekeu J, Maritan A and Stella A L 1982 *J. Phys. A: Math. Gen.* **15** L291–7
Kac M 1968 in *Statistical Physics, Phase Transitions and Superfluidity* ed M Chréfien, E P Gross and S Deser (New York: Gordon and Breach) vol 1
Kirkpatrick S 1976 *Phys. Rev. Lett.* **36** 69–72
Livi R, Maritan A, Ruffo S and Stella A L 1983 *Phys. Rev. Lett.* **50** 459–63
Nakanishi H and Stanley H E 1981 *J. Phys. A: Math. Gen.* **14** 593–720
Nienhuis B, Riedel E K and Schick M 1981 *Phys. Rev.* B **23** 6055–60
Pfeuty P and Toulouse G 1977 *Introduction to the Renormalization Group and to Critical Phenomena* (London: Wiley)
Potts R B 1952 *Proc. Camb. Phil. Soc.* **48** 106–9
Simon B 1980 *J. Stat. Phys.* **22** 491–3
Temperley H N V 1954 *Proc. R. Soc.* **67** 233
Toulouse G 1974 *Nuovo Cimento* B **23** 234–40
Wu F Y 1982 *Rev. Mod. Phys.* **54** 235–68